Electronic Companion to "Supply Chain Dynamics and Channel Efficiency in Durable Product Pricing and Distribution"

Wei-yu Kevin Chiang

College of Business, City University of Hong Kong

wchiang@cityu.edu.hk

I. Proof of Proposition 1 (The Optimal Pricing Strategy)

Based on (8), we obtain the following optimality conditions

$$\frac{\partial H(x,p)}{\partial p} = \alpha\Big(N - 2p - x + c\Big) - \alpha\lambda = 0, \tag{S1}$$

$$\dot{\lambda}(t) = \delta\lambda - \frac{\partial H(x,p)}{\partial x} = \lambda\Big(\alpha + \delta\Big) + \alpha(p - c), \tag{S2}$$

$$\dot{x}(t) = \alpha\Big(N - p - x\Big). \tag{S3}$$

From (S1) we have $p = \Big(N + c - x - \lambda\Big) / 2$, which when substituted into (S2) and (S3) gives two differential equations in terms of x and λ:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \mathbf{A}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \mathbf{b}\text{ , where } \mathbf{A} = \frac{a}{2}\begin{bmatrix} -1 & 1 \\ -1 & \frac{\alpha+2\delta}{\alpha} \end{bmatrix} \text{ and } \mathbf{b} = \frac{a(N-c)}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{S4}$$

The two eigenvalues of $\mathbf{A}$ are $r_1 = -\Big(\sqrt{2\alpha\delta + \delta^2} - \delta\Big) / 2$ and $r_2 = \Big(\sqrt{2\alpha\delta + \delta^2} + \delta\Big) / 2$. Define two new variables $u(t)$ and $v(t)$ as linear combinations of $x(t)$ and $\lambda(t)$:

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \mathbf{H}^{-1}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}\text{ , where } \mathbf{H} = \begin{bmatrix} \frac{\alpha+2r_2}{\alpha} & \frac{\alpha+2r_1}{\alpha} \\ 1 & 1 \end{bmatrix}. \tag{S5}$$

Note that each column in $\mathbf{H}$ is an eigenvector of $\mathbf{A}$. Then, we can transform (S4) into a diagonal system consisting of single-endogenous-variable differential equations:

$$\begin{bmatrix} \dot{u}(t) \\ \dot{v}(t) \end{bmatrix} = \mathbf{H}^{-1}\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \mathbf{H}^{-1}\mathbf{A}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \mathbf{H}^{-1}\mathbf{b} = \mathbf{H}^{-1}\mathbf{H}\mathbf{\Lambda}\mathbf{H}^{-1}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \mathbf{H}^{-1}\mathbf{b} = \mathbf{\Lambda}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \mathbf{H}^{-1}\mathbf{b}, \tag{S6}$$

where $\mathbf{\Lambda}$ is the 2×2 diagonal matrix whose diagonal elements are the two eigenvalues of $\mathbf{A}$. It is straightforward to obtain the following general solution for the transformed system in (S6):

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} - \mathbf{\Lambda}^{-1}\mathbf{H}^{-1}\mathbf{b}, \tag{S7}$$

where k_1 and k_2 are arbitrary constants to be determined. Substituting in (S5), we convert the solution back into the original variables $x(t)$ and $\lambda(t)$. That is,

$$\begin{aligned}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix} &= \mathbf{H}\begin{bmatrix}u(t)\\ v(t)\end{bmatrix} = \mathbf{H}\begin{bmatrix}e^{r_1 t} & 0\\ 0 & e^{r_2 t}\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix} - \mathbf{H}\boldsymbol{\Lambda}^{-1}\mathbf{H}^{-1}\mathbf{b} = \mathbf{H}\begin{bmatrix}e^{r_1 t} & 0\\ 0 & e^{r_2 t}\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix} - \mathbf{A}^{-1}\mathbf{b}\\ &= \begin{bmatrix}\frac{\alpha+2r_2}{\alpha} & \frac{\alpha+2r_1}{\alpha}\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{r_1 t} & 0\\ 0 & e^{r_2 t}\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix} - \begin{bmatrix}-\frac{2\delta+\alpha}{\alpha\delta} & \frac{1}{\delta}\\ -\frac{1}{\delta} & \frac{1}{\delta}\end{bmatrix}\begin{bmatrix}\frac{\alpha(N-c)}{2}\\ \frac{\alpha(N-c)}{2}\end{bmatrix}\\ &= \begin{bmatrix}\frac{\alpha+2r_2}{\alpha}e^{tr_1} & \frac{\alpha+2r_1}{\alpha}e^{tr_2}\\ e^{tr_1} & e^{tr_2}\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix} + \begin{bmatrix}N-c\\ 0\end{bmatrix}.\end{aligned} \tag{S8}$$

The boundary conditions $x(0) = 0$ and $\lim_{t\to\infty} e^{-\delta t}\lambda(t)x(t) = 0$ imply $k_1 = \frac{\alpha(N-c)}{-(\alpha+2r_2)}$ and $k_2 = 0$. Substituting in (S8), it follows that $x^F(t) = (N-c)(1-e^{-\gamma t})$ and $\lambda^F(t) = -(N-c)(1-2\gamma/\alpha)e^{-\gamma t}$, where $\gamma = -r_1$. Substituting in (S1) yields the optimal price path $p^F(t)$.

II. Proof of Proposition 4 (Myopic Equilibrium)

Plugging (21) into (22) yields $\pi_m^M(w) = (w-c)\alpha\left(N - \left(N+w-x\right)/2 - x\right)$. The first order condition of $\pi_m^M(w)$ implies $\tilde{w}^M = \left(N+c-x\right)/2$, which after substituting into (6) yields $\dot{x} = (\alpha/4)(N-c-x)$. Solving the differential equation with $x(0)=0$ yields (24). The result in (23) follows immediately after plugging (24) into $\tilde{w}^M$ above and then into (21).

III. Proof of Proposition 5 (Benefit from Myopic Pricing)

With (20) and (25), it can be verified that $\pi_m^{OL} - \pi_m^M = \left(\frac{\alpha+\delta-\sqrt{2\alpha\delta+\delta^2}}{4\alpha} - \frac{\alpha}{4\left(\alpha+2\delta\right)}\right)(N-c) =$

$\frac{3\alpha\delta+2\delta^2-\left(\alpha+2\delta\right)\sqrt{\delta\left(\delta+2\alpha\right)}}{4\alpha\left(\alpha+2\delta\right)}(N-c) = \frac{-2\alpha^3\delta(N-c)}{4\alpha\left(\alpha+2\delta\right)\left(3\alpha\delta+2\delta^2+\left(\alpha+2\delta\right)\sqrt{\delta\left(\delta+2\alpha\right)}\right)} < 0$. Similarly, we can verify $\pi_r^{OL} - \pi_r^M < 0$, $\pi_m^{FB} - \pi_m^M < 0$, and $\pi_r^{FB} - \pi_r^M < 0$. The result then follows. With (12) and (25), the condition $\alpha = 4\delta$ can be derived by equating $\pi_m^M + \pi_r^M$ to π^F, and then solving for α.

IV. Proof of Proposition 6 (Strategic Decentralization)

From (25) we have $\pi_m^M + \pi_r^M = \frac{3\alpha\left(N-c\right)^2}{8\left(\alpha+2\delta\right)}$, and from (13) we know $\pi^{\mathrm{M}} = \frac{\alpha(N-c)^2}{4(\alpha+\delta)}$. Equating π^M to $\pi_m^M + \pi_r^M$ and then solving for α result in $\alpha = \delta$, which concludes $\pi_m^M + \pi_r^M > \pi^M$ if $\alpha > \delta$.

V. Proof of Proposition 7 (Disintermediation Conditions)

When the forward-looking manufacturer sells directly to customers, it acts as a monopolist; thus according to (12), its net discounted profit with α_m, is given by

$$\left(\alpha_m + \delta - \sqrt{2\alpha_m\delta + \delta^2}\right)(N-c)^2 \,/\, (2\alpha_m). \tag{S9}$$

On the other hand, when selling through a forward-looking retailer with the trial α_r, based on Table 1(a) the forward-looking manufacturer will obtain the following profit

$$\left(\alpha_r + \delta - \sqrt{2\alpha_r\delta + \delta^2}\right)(N-c) \,/\, (4\alpha_r). \tag{S10}$$

By equating (S9) and (S10) and then solving for α_m we obtain $\theta^{OL}_{(F,F)} = \dfrac{4\alpha_r\delta}{5\delta + \alpha_r + 3\sqrt{2\alpha_r\delta + \delta^2}}$. Similarly, we can obtain the other thresholds in the case of open-loop equilibrium:

$$\theta^{OL}_{(M,F)} = \frac{2\alpha_r\delta^2}{2\delta^2 + \left(2\delta + \alpha_r\right)\sqrt{\alpha_r\delta + \delta^2}} \text{ and } \theta^{OL}_{(F,M)} = \theta_{(M,M)} = \alpha_r/2\,.$$

In the same vain, with (12) and Table 1(b), the following thresholds in the case of feedback equilibrium can be derived:

$$\theta^{FB}_{(F,F)} = 4\delta\alpha_r \frac{3\alpha_r + 52\delta - 10\sqrt{6\delta\alpha_r + 4\delta^2}}{\left(16\delta - \alpha_r\right)^2},\ \theta^{FB}_{(M,F)} = 4\delta\frac{6\left(\alpha_r + \delta\right)\sqrt{4\delta^2 + 2\delta\alpha_r} - \left(12\delta^2 + 3\delta\alpha_r - 2\alpha_r^{\,2}\right)}{96\delta^2 + 45\delta\alpha_r - 2\alpha_r^{\,2}},$$

and $\theta^{FB}_{(F,M)} = \dfrac{6\delta\alpha_r\left(\alpha_r + \delta\right)\left(2\delta^2 + 3\delta\alpha_r - 2\sqrt{\delta\left(\alpha_r + \delta\right)}\left(\alpha_r + \delta\right)\right)}{2\delta\left(\alpha_r + \delta\right)\left(2\delta - \alpha_r\right)\left(3\alpha_r + 2\delta\right) - \sqrt{\delta\left(\alpha_r + \delta\right)}\left(\alpha_r + 2\delta\right)^3}$.

The result $\theta^{OL}_{(F,F)} < \alpha_r/2$ can be verified by showing $\dfrac{\partial\theta^{OL}_{(F,F)}}{\partial\delta} = \dfrac{4\alpha^2\left(\sqrt{\delta^2 + 2\alpha\delta} + 3\delta\right)}{\left(\alpha + 5\delta + 3\sqrt{\delta^2 + 2\alpha\delta}\right)^2\sqrt{\delta^2 + 2\alpha\delta}} > 0$

and $\lim\limits_{\delta\to\infty}\theta^{OL}_{(F,F)} = \lim\limits_{\delta\to\infty}\dfrac{4\alpha_r}{5 + \alpha_r/\delta + 3\sqrt{2\alpha_r/\delta + 1}} = \dfrac{\alpha_r}{2}$. To verify $\theta^{OL}_{(F,F)} > \theta^{OL}_{(M,F)}$, since

$\theta^{OL}_{(F,F)} - \theta^{OL}_{(M,F)} = \dfrac{2\alpha\delta\left(2\left(2\delta + \alpha\right)\sqrt{\alpha\delta + \delta^2} - \delta^2 - \alpha\delta - 3\delta\sqrt{2\alpha\delta + \delta^2}\right)}{\left(\alpha + 5\delta + 3\sqrt{2\alpha\delta + \delta^2}\right)\left(2\delta^2 + \left(2\delta + \alpha\right)\sqrt{\alpha\delta + \delta^2}\right)}$, it suffices to show

$$2\left(2\delta + \alpha\right)\sqrt{\alpha\delta + \delta^2} > \delta^2 + \alpha\delta + 3\delta\sqrt{2\alpha\delta + \delta^2}\,. \tag{S11}$$

The difference between the left hand side and the right hand side of (S11), after squaring the items on both sides, is $6\delta^4 + \alpha\delta\left(3\delta + 4\alpha\right)\left(4\delta + \alpha\right) - 6\delta^2\left(\alpha + \delta\right)\sqrt{2\alpha\delta + \delta^2}$, which is positive. The rest of the results can be verified with the same approach.

VI. Optimality Conditions for the Numerical Study in Section 7.3

(i) *The Optimal Pricing*

The problem is to maximize (7) subject to (26), (27), (29), and (30). Accordingly, the current-value Lagrangian is given by $L(p,x,r,\lambda_1,\lambda_2,u,t)=(p(t)-c(t))\dot{x}(t)+\lambda_1(t)\dot{x}(t)+\lambda_2(t)\dot{r}(t)+u(K-\dot{x}(t))$, where $\lambda_1(t)$ and $\lambda_2(t)$ are the shadow prices associated with x and r, respectively, and the scalar $u>0$ is the Lagrange multiplier. The optimal pricing can be obtained by solving the following optimality conditions:

$$\frac{\partial L}{\partial p}=0\Rightarrow p=\frac{N-x+\Omega r}{2(1+\Omega)}-\frac{\lambda_1-u-(c_0+c_1e^{-\Lambda x})}{2}+\frac{\lambda_2\kappa}{2(1+\Omega)(\alpha+\beta x\,/\,N)}, \tag{S12}$$

$$\dot{x}(t)=\left(\alpha+\beta x\,/\,N\right)\left(N-x+\Omega r+(1+\Omega)\left(\lambda_1-u-(c_0+c_1e^{-\Lambda x})\right)\right)/2-\lambda_2\kappa/2, \tag{S13}$$

$$\dot{r}(t)=\frac{\kappa}{2}\left(\frac{N-x-(2+\Omega)r}{(1+\Omega)}-\lambda_1+u+(c_0+c_1e^{-\Lambda x})+\frac{\lambda_2\kappa}{(1+\Omega)(\alpha+\beta x\,/\,N)}\right), \tag{S14}$$

$$\begin{aligned}\dot{\lambda}_1=\delta\lambda_1+\lambda_2\kappa\Lambda c_1e^{-\Lambda x}-\frac{1}{2}\left(N-x+\Omega r+(1+\Omega)\left(\lambda_1-u-(c_0+c_1e^{-\Lambda x})\right)+\frac{\lambda_2\kappa}{\alpha+\beta x/N}\right)\left(\beta\frac{N-x+\Omega r}{2N(1+\Omega)}\right.\\ \left.+\beta\frac{\lambda_1-u-(c_0+c_1e^{-\Lambda x})}{2N}-\frac{\beta\lambda_2\kappa}{2N(1+\Omega)(\alpha+\beta x/N)}-\frac{\alpha+\beta x\,/\,N}{1+\Omega}+\Lambda c_1e^{-\Lambda x}(\alpha+\beta x/N)\right),\end{aligned} \tag{S15}$$

$$\dot{\lambda}_2=(\delta+\kappa)\lambda_2-\frac{(\alpha+\beta x\,/\,N)\Omega}{2}\left(\frac{N-x+\Omega r}{1+\Omega}+\lambda_1-u-(c_0+c_1e^{-\Lambda x})+\frac{\lambda_2\kappa}{(1+\Omega)(\alpha+\beta x\,/\,N)}\right), \tag{S16}$$

$$u\left(K-\left(\alpha+\beta x\,/\,N\right)\left(N-x+\Omega r+(1+\Omega)\left(\lambda_1-u-(c_0+c_1e^{-\Lambda x})\right)\right)/2+\lambda_2\kappa/2\right)=0. \tag{S17}$$

(ii) *Myopic Pricing in the Decentralized Supply Chain*

When the manufacturer and the retailer are myopic, they maximize their respective current-term profits

$\pi_m=(w(t)-c(t))\dot{x}(t)$ and $\pi_r=(p(t)-w(t))\dot{x}(t)$, subject to (26), (27), (29), and (30).

Given the wholesale price w, the best price reaction for the retailer is

$$\frac{\partial \pi_r}{\partial p}=0\Rightarrow p=\frac{N-x+\Omega r}{2(1+\Omega)}+\frac{w}{2}, \tag{S18}$$

which, after plugging into (29) and (30), yields the following sales rate and reference price rate:

$$\dot{x}(t)=(\alpha+\beta x\,/\,N)\left(N-x+\Omega r-(1+\Omega)w\right)/2, \tag{S19}$$

$$\dot{r}(t)=\kappa\left(\frac{N-x-(2+\Omega)r}{2(1+\Omega)}+\frac{w}{2}\right). \tag{S20}$$

Subject to (S19), (S20), and (27), the Lagrangian for the manufacturer's optimization problem is given by $L(w,x,u,t)=(w(t)-c(t))\dot{x}(t)+u(K-\dot{x}(t))$, where the scalar $u>0$ is the Lagrange multiplier. Accordingly, the myopic equilibrium pricing corresponds to the solution of the following optimality conditions:

$$\frac{\partial \pi_r}{\partial p} = 0 \Rightarrow p = \frac{N - x + \Omega r}{2(1+\Omega)} + \frac{w}{2}, \tag{S21}$$

$$\dot{x}(t) = (\alpha + \beta x / N)\Big(N - x + \Omega r - (1+\Omega)u - (1+\Omega)(c_0 + c_1 e^{-\Lambda x})\Big)/4, \tag{S22}$$

$$\dot{r}(t) = \frac{\kappa}{4(1+\Omega)}\Big(3\big(N - x\big) - (4+\Omega)r + (1+\Omega)\Big(u + (c_0 + c_1 e^{-\Lambda x})\Big)\Big), \tag{S23}$$

$$u\Big(K - (\alpha + \beta x / N)\Big(N - x + \Omega r - (1+\Omega)u - (1+\Omega)(c_0 + c_1 e^{-\Lambda x})\Big)/4\Big) = 0. \tag{S24}$$

VII. Computational Result of the Numerical Study in Section 7

	Cost Learning Effect:	*Absent* (Λ=0)			*Fair* (Λ=0.05)			*High* (Λ=0.10)		
		Reference Price Effect			**Reference Price Effect**			**Reference Price Effect**		
Imitation Effect	**Discount Rate**	*Absent* (Ω=0)	*Fair* (Ω=0.25)	*High* (Ω=0.50)	*Absent* (Ω=0)	*Fair* (Ω=0.25)	*High* (Ω=0.50)	*Absent* (Ω=0)	*Fair* (Ω=0.25)	*High* (Ω=0.50)
					No Capacity Constraint (K= ∞)					
Absent (β=0)	*Low* (δ=0.05)	98.17%	99.66%	99.97%	96.52%	98.83%	99.76%	96.90%	98.82%	99.73%
	Fair (δ=0.10)	93.29%	96.27%	98.03%	90.03%	93.67%	95.93%	90.42%	94.06%	96.15%
	High (δ=0.15)	89.81%	93.05%	95.15%	85.93%	89.63%	92.05%	86.00%	89.83%	92.20%
Fair (β=0)	*Low* (δ=0.05)	98.44%	99.81%	99.96%	96.85%	98.96%	99.74%	97.06%	98.12%	99.38%
	Fair (δ=0.10)	93.07%	96.25%	98.04%	89.61%	93.48%	95.78%	90.09%	93.86%	96.11%
	High (δ=0.15)	89.08%	92.61%	94.86%	84.67%	88.73%	91.29%	84.94%	88.96%	91.61%
High (β=0)	*Low* (δ=0.05)	98.71%	99.83%	99.84%	97.15%	98.99%	99.58%	97.44%	99.10%	99.58%
	Fair (δ=0.10)	93.30%	96.49%	98.27%	89.73%	93.59%	95.88%	90.24%	94.02%	96.23%
	High (δ=0.15)	88.81%	92.53%	95.04%	84.15%	88.29%	90.99%	84.46%	88.64%	91.36%
					Fair Capacity Constraint (K=4)					
Absent (β=0)	*Low* (δ=0.05)	98.14%	99.69%	99.98%	96.53%	98.82%	99.76%	96.73%	98.82%	99.73%
	Fair (δ=0.10)	93.21%	96.35%	98.06%	90.08%	93.73%	95.95%	90.41%	93.93%	96.22%
	High (δ=0.15)	89.69%	93.22%	95.31%	85.98%	88.69%	92.10%	86.66%	89.81%	92.26%
Fair (β=0)	*Low* (δ=0.05)	98.47%	99.82%	99.96%	97.15%	98.96%	99.74%	97.20%	99.09%	99.72%
	Fair (δ=0.10)	93.16%	96.32%	98.06%	89.66%	93.45%	95.77%	89.96%	93.86%	96.08%
	High (δ=0.15)	89.21%	91.93%	94.89%	84.78%	88.67%	90.20%	84.87%	88.77%	91.69%
High (β=0)	*Low* (δ=0.05)	98.71%	99.83%	99.81%	97.15%	99.01%	99.62%	97.28%	99.10%	99.59%
	Fair (δ=0.10)	93.31%	96.49%	98.19%	89.72%	93.38%	95.78%	90.17%	93.89%	96.46%
	High (δ=0.15)	88.96%	92.60%	94.78%	84.19%	88.53%	92.04%	84.37%	89.30%	93.12%
					High Capacity Constraint (K=3)					
Absent (β=0)	*Low* (δ=0.05)	98.19%	99.68%	99.96%	96.53%	98.82%	99.76%	96.73%	98.95%	99.77%
	Fair (δ=0.10)	93.33%	96.34%	98.05%	89.78%	93.67%	96.07%	90.36%	94.24%	96.76%
	High (δ=0.15)	83.67%	92.62%	94.80%	86.06%	90.48%	93.72%	86.77%	91.34%	94.88%
Fair (β=0)	*Low* (δ=0.05)	98.48%	99.81%	99.95%	96.86%	98.96%	99.56%	96.86%	98.90%	99.93%
	Fair (δ=0.10)	91.53%	96.28%	97.75%	90.05%	94.58%	97.76%	91.04%	95.61%	98.83%
	High (δ=0.15)	88.77%	92.50%	95.04%	86.40%	91.98%	96.37%	87.63%	93.44%	97.85%
High (β=0.2)	*Low* (δ=0.05)	98.71%	99.83%	99.81%	96.98%	99.23%	99.89%	97.28%	99.46%	99.84%
	Fair (δ=0.10)	92.73%	96.14%	98.14%	91.77%	97.03%	98.82%	93.05%	97.73%	99.12%
	High (δ=0.15)	88.79%	93.28%	96.37%	88.48%	95.09%	97.95%	90.20%	91.20%	98.43%

Note that the shaded area in the upper left corner of the table corresponds to the analytical results in Section 5, where all additional effects are absent.